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THE INVERSE PROBLEM FOR SMALL RANDOM PERTURBATIONS OF DYNAMICAL SYSTEMS

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ABSTRACT
What can one get to know about the dynamical system from its small random perturbation? What can one say about solutions of an ordinary differential equation $\dot{x}_t = B(x_t)$ having some information on its singular perturbation operator $L^\varepsilon = \varepsilon L + (B, \nabla)$ with L being an elliptic second order operator? These problems are studied in the paper.

1. Introduction

Let S^t be a continuous in t group of homeomorphisms (the flow) of a complete metric locally compact space X with t being a continuous parameter $-\infty < t < \infty$ or a discrete one $t = \cdots, -2, -1, 0, 1, 2, \cdots$.

Consider a family of time-homogeneous Markov processes x_t^ε in X with continuous trajectories, where ε is a positive parameter. Let $P^\varepsilon(t, x, V)$ be the transition probability of x_t^ε and suppose that all x_t^ε satisfy the Feller property (see [2]), i.e., for any continuous function f on X the function $P_t^\varepsilon f$ defined by

$$P_t^\varepsilon f(x) = \int_X P^\varepsilon(t, x, dy) f(y) \tag{1.1}$$

is also continuous.

The family x_t^ε is called a small random perturbation of the dynamical system S^t if for any continuous f and all $t > 0$,

$$\sup_{x \in X} |P_t^\varepsilon f(x) - f(S^t x)| \to 0 \qquad \text{as } \varepsilon \to 0. \tag{1.2}$$

We distinguish two general problems about small random perturbations of dynamical systems. The first one is the direct problem, i.e., the study of

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asymptotic behaviour as $\varepsilon \to 0$ of parameters of the process x_t^ε provided some information about the dynamical system S^t and the type of perturbations are known. The second one is the inverse problem, i.e., the study of the dynamical system S^t when the asymptotic behaviour as $\varepsilon \to 0$ of some probabilistic parameters of the processes x_t^ε is known.

Most papers on this matter deal with the direct problem (see, for instance, [2] and [7]). Some papers can be considered as dealing with both kinds of problems (see [4] and [6]). An example of the inverse problem is given by the Hasminskii's lemma (see [3]). It claims that if μ^ε is an invariant measure of the process x_t^ε and for some sequence $\varepsilon_i \to 0$ there is a weak limit

$$\text{(w)} \lim_{\varepsilon_i \to 0} \mu^{\varepsilon_i} = \mu \tag{1.3}$$

then μ is an invariant measure of the dynamical system S^t.

In this paper we consider other examples of the inverse problem which have applications to the elliptic singular perturbation problem in the theory of partial differential equations.

Let there be given a bounded connected open set $G \subset X$ with the boundary ∂G. Denote by τ the exit time from G, i.e.,

$$\tau = \inf\{t \geqq 0 : x_t^\varepsilon \notin G\}. \tag{1.4}$$

As usual, denote by $P_x^\varepsilon\{A\}$ and $E_x^\varepsilon \xi$ the probability of the event A and the expectation of the random value ξ, respectively, for the process x_t^ε starting at x.

We shall prove that the asymptotic behaviour as $t \to \infty$ and $\varepsilon \to 0$ of $P_x^\varepsilon\{\tau > t\}$ and $E_x^\varepsilon \tau$ determines whether the dynamical system S^t has a closed invariant set in $G \cup \partial G$ or not. Under some conditions the asymptotic behaviour of $P_x^\varepsilon\{\tau > t\}$ is determined by the asymptotics of the principal eigenvalue of the operator P_t^ε defined by (1.1).

When x_t^ε is the diffusion perturbation of the smooth dynamical system S^t then we shall find the connection between the existence of the invariant set of the dynamical system S^t in $G \cup \partial G$ and the asymptotic behaviour of the principal eigenvalue and the solution of the Poisson boundary value problem for the corresponding elliptic singularly perturbed operator.

2. The general case

A set $\Lambda(G) \subset G \cup \partial G$ is called the maximal invariant (under the action of S^t) set in $G \cup \partial G$ if any set Ω satisfying the property

(2.1) $S^t\Omega = \Omega \subset G \cup \partial G$ for all $t \in (-\infty, \infty)$ is a subset of $\Lambda(G)$,

i.e., $\Omega \subset \Lambda(G)$. Obviously, $\Lambda(G)$ is a closed set (maybe empty).

The main result is the following theorem.

THEOREM 2.1. (a) *If for some* $x \in G$,

$$\limsup_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \ln P_x^\varepsilon\{\tau > t\} > -\infty, \tag{2.2}$$

then the maximal invariant set $\Lambda(G)$ *is not empty*;

(b) *If for some* $x \in G$,

$$\limsup_{\varepsilon \to 0} E_x^\varepsilon \tau = \infty, \tag{2.3}$$

then the set $\Lambda(G)$ *is not empty.*

PROOF. Suppose that $\Lambda(G)$ is empty. Then there is an open bounded domair $\tilde{G} \supset G \cup \partial G$ such that the corresponding set $\Lambda(\tilde{G})$ is also empty.

Indeed, it is easy to see that if $G_1 \subset G_2$ then $\Lambda(G_1) \subset \Lambda(G_2)$. Take the sequence $\{G_m\}$ of open domains such that

$$G_1 \supset G_2 \supset G_3 \cdots \quad \text{and} \quad \bigcap_{m \geqq 1} G_m = G \cup \partial G. \tag{2.4}$$

If $\Lambda(G_m) \neq \emptyset$ for all $m \geqq 1$ then

$$\Lambda = \bigcap_{m \geqq 1} \Lambda(G_m) \neq \emptyset \tag{2.5}$$

since $\Lambda(G_m)$ is closed and $\Lambda(G_1) \supset \Lambda(G_2) \supset \Lambda(G_3) \supset \cdots$.

It is clear that Λ is invariant with respect to S^t and $\Lambda \subset G \cup \partial G$. Thus $\phi \neq \Lambda \subset \Lambda(G)$. This contradicts our assumption and we conclude that for some $m_0 \geqq 1$,

$$\Lambda(G_{m_0}) = \emptyset.$$

It remains to set $\tilde{G} = G_{m_0}$.

For any $x \in \tilde{G} \cup \partial\tilde{G}$ put

$$t(x) = \inf\{t \geqq 0 : S^t x \notin \tilde{G} \cup \partial\tilde{G}\}. \tag{2.6}$$

Let us prove that

$$t(x) < \infty \qquad \text{for each } x \in \tilde{G} \cup \partial\tilde{G}. \tag{2.7}$$

Indeed, if $t(x_0)=\infty$ then $S^t x_0 \in \tilde{G} \cup \partial \tilde{G}$ for all $t \geqq 0$. Thus for some sequence $t_n \uparrow \infty$ and a point $y \in \tilde{G} \cup \partial \tilde{G}$ one has

$$S^{t_n} x_0 \to y \qquad \text{as } t_n \uparrow \infty.$$

Then also

$$S^{t_n + t} x_0 \to S^t y \qquad \text{as } t_n \uparrow \infty \quad \text{for any } t \in (-\infty, \infty).$$

One easily obtains from here that $S^t y \in \tilde{G} \cup \partial \tilde{G}$ for all $t \in (-\infty, \infty)$ and so the set $\{S^t y, t \in (-\infty, \infty)\} \subset \Lambda(\tilde{G}) = \varnothing$. This contradiction proves (2.7).

From the upper semicontinuity of $t(x)$ we conclude that

$$T = \sup_{x \in \tilde{G} \cup \partial \tilde{G}} t(x) < \infty. \tag{2.8}$$

Notice that (1.2) implies the following condition:

$$\limsup_{\varepsilon \to 0} \sup_{x \in G} P_x^\varepsilon\{\operatorname{dist}(x_t^\varepsilon, S^t x) > \delta\} = 0 \qquad \text{for any } \delta > 0 \quad \text{and} \quad t > 0. \tag{2.9}$$

Now by (2.6)–(2.9) one easily obtains that

$$\rho(\varepsilon) = \sup_{x \in G} P_x^\varepsilon\{\tau > T\} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.10}$$

By the Markov property (see [2]),

$$\begin{aligned} P_x^\varepsilon\{\tau > mT\} &= E_x^\varepsilon \chi_{\tau > T} E_{x_T^\varepsilon}^\varepsilon \chi_{\tau > T} \cdots E_{x_T^\varepsilon} \chi_{\tau > T} \\ &\leqq [\rho(\varepsilon)]^m, \end{aligned} \tag{2.11}$$

where χ_A is the indicator of the event A.

Since $P_x^\varepsilon\{\tau > t\}$ decreases in t then we get

$$Q_x^\varepsilon = \limsup_{t \to \infty} \frac{1}{t} \ln P_x^\varepsilon\{\tau > t\} \leqq \frac{1}{T} \ln \rho(\varepsilon). \tag{2.12}$$

Taking into account (2.10) one concludes by (2.12) that for any $x \in G$,

$$Q_x^\varepsilon \to -\infty \qquad \text{as } \varepsilon \to 0 \tag{2.13}$$

that contradicts (2.2). Thus our assumption that $\Lambda(G) = \varnothing$ is inconsistent.

To prove the item (b) notice that

$$E_x^\varepsilon \tau \leqq T \sum_{m=1}^{\infty} P_x^\varepsilon\{\tau > (m-1)T\}. \tag{2.14}$$

Assumption $\Lambda(G)=\emptyset$ gives, in view of (2.10), (2.11) and (2.14) for ε small enough and $x \in G$, that

$$E_x^\varepsilon \tau \leqq T \sum_{m=1}^{\infty} [\rho(\varepsilon)]^{m-1} < \infty. \tag{2.15}$$

This contradiction with (2.3) completes the proof of Theorem 2.1.

3. Diffusion perturbation and application to the elliptic singular perturbation problem

Let X be the Euclidean space R^n and x_t^ε be a diffusion process in R^n generated by the elliptic differential operator of the second order

$$L^\varepsilon = \varepsilon^2 L + (B, \nabla), \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \tag{3.1}$$

with smooth coefficients, where L is a nondegenerate elliptic operator and

$$\frac{d(S^t x)}{dt} = B(S^t x). \tag{3.2}$$

Assume that the bounded domain G has smooth boundary ∂G. Under these circumstances, by lemma 3.1 of [5] one obtains that for any $x \in G$ there is a limit

$$\lambda^\varepsilon = \lim_{t\to\infty} \frac{1}{t} \ln P_x^\varepsilon\{\tau > t\}, \tag{3.3}$$

where τ is defined by (1.4) and λ^ε turns out to be the principal eigenvalue of the Dirichlet problem for the operator L^ε in the domain G.

Under assumptions of this paragraph we can improve the first part of Theorem 2.1 in the following way.

THEOREM 3.1. *The dynamical system S^t has no invariant set in $G \cup \partial G$, i.e., $\Lambda(G)=\emptyset$, if and only if*

$$\lim_{\varepsilon\to 0} \lambda^\varepsilon = -\infty. \tag{3.4}$$

PROOF. If $\Lambda(G)=\emptyset$ then by Theorem 2.1 we get $\limsup_{\varepsilon\to 0} \lambda^\varepsilon = -\infty$ and so $\lim_{\varepsilon\to 0} \lambda^\varepsilon = -\infty$. This follows also from the paper [8].

Suppose now that (3.4) is true and prove that $\Lambda(G)=\emptyset$. To do this we assume that $\Lambda(G)\neq\emptyset$ and prove that

$$\limsup_{\varepsilon\to 0} \lambda^\varepsilon > -\infty. \tag{3.5}$$

We shall see that, in fact, also

$$\liminf_{\varepsilon\to 0} \lambda^{\varepsilon} > -\infty, \tag{3.6}$$

if we assume $\Lambda(G) \neq \emptyset$.

Set $G_\rho = \{x \in G : \mathrm{dist}(x, \partial G) > \rho\}$, $U_\rho(z) = \{y : |z - y| < \rho\}$, and denote by $\mathcal{O}$ the origin.

Let $x \in \Lambda(G)$, i.e., $S^t x \in G \cup \partial G$ for all t. It is easy to see that one can find positive numbers $\varepsilon_0, \delta, r_2 < r_1 < 1$ and some points $Q_m \in U_2(\mathcal{O})$, $m = 0, 1, 2, \cdots$, such that the balls $V_m^{(1)} = U_{r_1}(Q_m)$ and $V_m^{(2)} = U_{r_2}(Q_m)$ satisfy for all $m = 0, 1, 2, \cdots$ the following properties:

(i) $U_2(\mathcal{O}) \supset V_m^{(1)} \supset (V_m^{(2)} \cup \partial V_m^{(2)}) \cup (V_{m+1}^{(2)} \cup \partial V_{m+1}^{(2)})$;

(ii) $\varepsilon V_m^{(1)} + S^{m\delta+t} x \subset G$ for all $t \in [0, \delta]$ and all $\varepsilon \in [0, \varepsilon_0]$;

(iii) $\varepsilon V_m^{(2)} + S^{m\delta} x \subset G_\varepsilon \cap U_{2\varepsilon}(S^{m\delta} x)$ for all $\varepsilon \in [0, \varepsilon_0]$.

The main step in the proof of Theorem 3.1 is the following result.

Lemma 3.1. *There is $q > 0$ such that for all $m = 0, 1, 2, \cdots$ one has*

$$\inf_{z \in \varepsilon V_m^{(2)} + S^{m\delta} x} P_z^{\varepsilon}\{x_\delta^{\varepsilon} \in \varepsilon V_{m+1}^{(2)} + S^{(m+1)\delta} x \text{ and } \tau > \delta\} \geqq q, \tag{3.7}$$

provided $0 < \varepsilon \leqq \varepsilon_0$, where, recall, τ is the exit time from G for the process x_t^ε.

Proof. Let $A(z)$ be the matrix of coefficients of the operator L in second order derivatives and $b(z)$ be the vector of coefficients in first order derivatives, i.e.,

$$L = \tfrac{1}{2}(A(z)\nabla, \nabla) + (b(z), \nabla).$$

The process x_t^ε satisfies the following stochastic differential equation:

$$dx_t^\varepsilon = \varepsilon \sigma(x_t^\varepsilon) dw_t + (\varepsilon^2 b(x_t^\varepsilon) + B(x_t^\varepsilon)) dt, \tag{3.8}$$

where $\sigma(z)\sigma^*(z) = A(z)$ and $w_t = (w_t^1, \cdots, w_t^n)$ is the Wiener process.

For any $m = 0, 1, 2, \cdots$ set $y_m^\varepsilon(t) = x_t^\varepsilon - S^{m\delta+t} x$. Let $z_m^\varepsilon(t)$ be the solution of the stochastic differential equation

$$dz_m^\varepsilon(t) = \varepsilon \sigma(z_m^\varepsilon(t) + S^{m\delta+t} x) dw_t + \varepsilon^2 b(z_m^\varepsilon(t) + S^{m\delta+t} x) dt. \tag{3.9}$$

The processes $y_m^\varepsilon(t)$ and $z_m^\varepsilon(t)$ differ just in drift, hence by Girsanov's formula (see, chapter 7 of [2]) and the properties of the domains $V_m^{(1)}$ and $V_m^{(2)}$ one gets

$$
\begin{aligned}
&P_z^\varepsilon\{x_\delta^\varepsilon \in \varepsilon V_{m+1}^{(2)} + S^{(m+1)\delta}x \text{ and } \tau > \delta\}\\
&\geqq P_z^\varepsilon\{x_\delta^\varepsilon \in \varepsilon V_{m+1}^{(2)} + S^{(m+1)\delta}x \text{ and } x_t^\varepsilon \in \varepsilon V_m^{(1)} + S^{m\delta+t}x \text{ for all } t\in[0,\delta]\}\\
(3.10)\qquad &= P_{z_m,0}\{y_m^\varepsilon(\delta) \in \varepsilon V_{m+1}^{(2)} \text{ and } y_m^\varepsilon(t)\in \varepsilon V_m^{(1)} \text{ for all } t\in[0,\delta]\}\\
&= E_{z_m,0}^\varepsilon \chi_{z_m^\varepsilon(\delta)\in\varepsilon V_{m+1}^{(2)}}\, \chi_{z_m^\varepsilon(t)\in\varepsilon V_m^{(1)} \text{ for all } t\in[0,\delta]} \frac{d\mu_{y_m^\varepsilon}}{d\mu_{z_m^\varepsilon}}(z_m^\varepsilon)\\
&= E_{z_m/\varepsilon,0}^\varepsilon \chi_{v_m^\varepsilon(\delta)\in V_{m+1}^{(2)}}\, \chi_{v_m^\varepsilon(t)\in V_m^{(1)} \text{ for all } t\in[0,\delta]} \frac{d\mu_{y_m^\varepsilon}}{d\mu_{z_m^\varepsilon}}(\varepsilon v_m^\varepsilon),
\end{aligned}
$$

where $z_m = z - S^{m\delta}x$, $v_m^\varepsilon(t) = z_m^\varepsilon/\varepsilon$, $P_{y,0}^\varepsilon$ and $E_{y,0}^\varepsilon$ denote the probability and the expectation of nonhomogeneous processes starting in zero time at y, and

$$
(3.11)\qquad \frac{d\mu_{y_m^\varepsilon}}{d\mu_{z_m^\varepsilon}}(\varepsilon v_m^\varepsilon) = \exp\left\{\frac{1}{\varepsilon}\int_0^\delta \sum_{i=1}^n h_i(t,\omega)dw_t^i - \frac{1}{2\varepsilon^2}\int_0^\delta \sum_{i=1}^n h_i^2(r,\omega)dt\right\},
$$

where $h(t,\omega) = (h_1(t,\omega),\cdots,h_n(t,\omega))$ defined by the formula

$$
h(t,\omega) = \sigma^{-1}(\varepsilon v_m^\varepsilon(t) + S^{m\delta+t}x)(B(\varepsilon v_m^\varepsilon(t) + S^{m\delta+t}x) - B(S^{m\delta+t}x)).
$$

Let θ_m be the exit time for the process $v_m^\varepsilon(t)$ from $V_m^{(1)}$; then (see [2])

$$
\begin{aligned}
E_v^\varepsilon \chi_{\theta_m>\delta}\left(\int_0^\delta \sum_{i=1}^n h_i(t,\omega)dw_t^i\right)^2 &\leqq E_v^\varepsilon\left(\int_0^{\min(\delta,\theta_m)} \sum_{i=1}^n h_i(t,\omega)dw_t^i\right)^2\\
&= E_v^\varepsilon \int_0^{\min(\delta,\theta_m)} \sum_{i=1}^n h_i^2(t,\omega)dt.
\end{aligned}
$$

Therefore by (3.11) and Chebyshev's inequality we easily get that for any $v\in V_m^{(2)}$ and each $a>0$,

$$
\begin{aligned}
&P_{v,0}^\varepsilon\left\{v_m^\varepsilon(t)\in V_m^{(1)} \text{ for all } t\in[0,\delta] \text{ and } \frac{d\mu_{y_m^\varepsilon}}{d\mu_{z_m^\varepsilon}}(\varepsilon v_m^\varepsilon) < e^{-a}\right\}\\
(3.12)\qquad &= P_{v,0}^\varepsilon\left\{\theta_m > \delta \text{ and } \left(\frac{1}{2\varepsilon^2}\int_0^\delta \sum_{i=1}^n h_i^2(t,\omega)dt - \frac{1}{\varepsilon}\int_0^\delta \sum_{i=1}^n h_i(t,\omega)dw_t^i\right) > a\right\}\\
&\leqq \frac{M}{a^2},
\end{aligned}
$$

where M depends just on the upper bounds of the norm $\|\sigma^{-1}(z)\|$ and the derivatives of the vector function $B(z)$, but not on ε and a.

On the other hand, by the definition of $v_m^\varepsilon(t)$ and the properties of the domains $V_m^{(1)}$ and $V_m^{(2)}$ one can easily see from the uniform estimates from below of the fundamental solution of the parabolic equation in a bounded domain given in theorem 8 of [1] that if $v \in V_m^{(2)}$ then

$$P_{v,0}^{\varepsilon}\{v_n^{\varepsilon}(\delta) \in V_{m+1}^{(2)} \text{ and } v_m^{\varepsilon}(t) \in V_m^{(1)} \text{ for all } t \in [0,\delta]\} \geqq \rho > 0, \tag{3.13}$$

for some constant $\rho > 0$ independent of m, ε and $v \in V_m^{(2)}$. Taking $z \in \varepsilon V_m^{(2)} + S^{m\delta}x$ in (3.10) we obtain that $z_m/\varepsilon \in V_m^{(2)}$ in (3.10). Therefore setting $a = 2M/\rho$ one gets (3.7) by (3.10), (3.12) and (3.13) with $q = \frac{1}{2}\rho \exp(-\sqrt{2M/\rho})$.

Now we can complete the proof of Theorem 3.1. By the Markov property for $z \in \varepsilon V_0^{(2)} + x$ one has

$$\begin{aligned} &P_z^{\varepsilon}\{\tau > k\delta\} \\ &= E_z^{\varepsilon}\chi_{\tau>\delta}E_{x_\delta^\varepsilon}^{\varepsilon}\chi_{\tau>\delta}\cdots E_{x_\delta^\varepsilon}^{\varepsilon}\chi_{\tau>\delta} \\ &\geqq E_z^{\varepsilon}\chi_{\tau>\delta}\chi_{x_\delta^\varepsilon \in \varepsilon V_1^{(2)} + S^\delta x}E_{x_\delta^\varepsilon}^{\varepsilon}\chi_{\tau>\delta}\chi_{x_\delta^\varepsilon \in \varepsilon V_2^{(2)} + S^{2\delta}x}\cdots E_{x_\delta^\varepsilon}^{\varepsilon}\chi_{\tau>\delta}\chi_{x_\delta^\varepsilon \in \varepsilon V_k^{(2)} + S^{k\delta}x} \\ &\geqq \left[\inf_{m=0,1,\cdots}\ \inf_{z \in \varepsilon V_m^{(2)} + S^{m\delta}x} P_z^{\varepsilon}\left\{x_\delta^\varepsilon \in \varepsilon V_{m+1}^{(2)} + S^{(m+1)\delta}x \text{ and } \tau > \delta\right\}\right]^k \\ &\geqq q^k. \end{aligned} \tag{3.14}$$

Hence, by (3.3) for all $\varepsilon \leqq \varepsilon_0$,

$$\lambda^{\varepsilon} \geqq \frac{1}{\delta}\ln q \tag{3.15}$$

that contradicts (3.4) and proves Theorem 3.1.

Notice that the function $u^{\varepsilon}(x) = E_x^{\varepsilon}\tau$ is the solution of the problem (see [2])

$$L^{\varepsilon}u^{\varepsilon} = -1, \qquad u^{\varepsilon}\big|_{\partial G} = 0. \tag{3.16}$$

The second part of Theorem 2.1 can be amplified as follows.

THEOREM 3.2. (a) *If for some* $x \in G$,

$$\limsup_{\varepsilon \to 0} u^{\varepsilon}(x) = \infty,$$

then the dynamical system S^t *has a nonempty invariant set* $\Lambda(G) \subset G \cup \partial G$;

(b) *If for any* $x \in G$,

$$\liminf_{\varepsilon \to 0} u^{\varepsilon}(x) < \infty, \tag{3.17}$$

then the open domain G *contains no invariant with respect to* S^t *closed subset*;

(c) *The item* (b) *cannot be improved, i.e., the case when for any* $x \in G$

(3.18) $$\limsup_{\varepsilon\to 0} u^{\varepsilon}(x) < \infty$$

and $\Lambda(G) \neq \varnothing$ *is possible.*

PROOF. The item (a) follows immediately from the item (b) of Theorem 2.1.

Suppose now that G contains an invariant closed subset Λ. Then there is positive $\delta = \inf_{x\in\Lambda} \operatorname{dist}(x, \partial G)$ and (2.9) is also true. Therefore one can easily get from [7] that for any $x \in \Lambda$ and $N > 0$

(3.19) $$P_x^{\varepsilon}\{\tau < N\} \to 0 \qquad \text{as } \varepsilon \to 0.$$

But

(3.20) $$E_x^{\varepsilon}\tau \geqq \sum_{m=1}^{\infty} (m-1) P_x^{\varepsilon}\{(m-1) \leqq \tau < m\} \geqq N P_x^{\varepsilon}\{\tau \geqq N\}.$$

Thus by (3.19),

$$\liminf_{\varepsilon\to 0} E_x^{\varepsilon}\tau \geqq N$$

for any N and so

$$\liminf_{\varepsilon\to 0} E_x^{\varepsilon}\tau = \infty$$

that contradicts (3.17) and proves the item (b) of Theorem 3.2.

Now let us describe the example which satisfies (3.18) and $\Lambda(G) \neq \varnothing$. Let $n = 2$,

(3.21) $$L^{\varepsilon} = \frac{1}{2}\varepsilon^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$$

and $G = \{(x_1, x_2) : (x_1 - 1)^2 + x_2^2 < 1\}$.

In this case $\Lambda(G)$ is the one point $(0, 0)$ and for any $x = (x_1, x_2)$,

(3.22) $$S^t x = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Therefore for each $x \in G$ there is $\tilde{t}(x) \in [\pi/2, 2\pi]$ such that $S^{\tilde{t}(x)} x \notin G \cup \partial G$. Let $p^{\varepsilon}(t, x, y)$ be the transition density of the process x_t^{ε} generated by L^{ε} in the whole plane R^2. Then

(3.23) $$p^{\varepsilon}(t, x, y) = \frac{1}{2\pi\varepsilon^2} \exp\left\{-\frac{1}{2\varepsilon^2} |y - S^t x|^2\right\}.$$

If $y \in U_\varepsilon(S^{\bar{t}(x)}x)$, then by (3.23)

$$p^\varepsilon(t,x,y) \geqq (2\pi\sqrt{e}\,\varepsilon^2)^{-1}. \tag{3.24}$$

The area of $U_\varepsilon(S^{\bar{t}(x)}x) \cap (R^2 \setminus G)$ is greater than $\pi\varepsilon^2/2$ and by (3.24) one gets

$$\begin{aligned} P_x^\varepsilon\{\tau \leqq 2\pi\} &\geqq P_x^\varepsilon\{\tau \leqq \bar{t}(x)\} \\ &\geqq P_x^\varepsilon\{x^\varepsilon_{\bar{t}(x)} \notin G\} \\ &\geqq P_x^\varepsilon\{x^\varepsilon_{\bar{t}(x)} \in U_\varepsilon(S^{\bar{t}(x)}x) \cap (R^2 \setminus G)\} \\ &\geqq (4\sqrt{e})^{-1}. \end{aligned} \tag{3.25}$$

Thus $P_x^\varepsilon\{\tau > 2\pi\} \leqq 1 - (4\sqrt{e})^{-1}$ and using the Markov property in the same way as in (2.11) we obtain

$$P_x^\varepsilon\{\tau > 2\pi m\} \leqq (1 - (4\sqrt{e})^{-1})^m. \tag{3.26}$$

Finally,

$$E_x^\varepsilon \tau \leqq 2\pi \sum_{m=1}^{\infty} P_x^\varepsilon\{\tau > 2\pi(m-1)\} \leqq 8\pi\sqrt{e} < \infty$$

that gives (3.18) in spite of $\Lambda(G) = \{(0,0)\} \neq \emptyset$.

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